A Short Guide to Hackenbush

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1 Let's Play A Game!

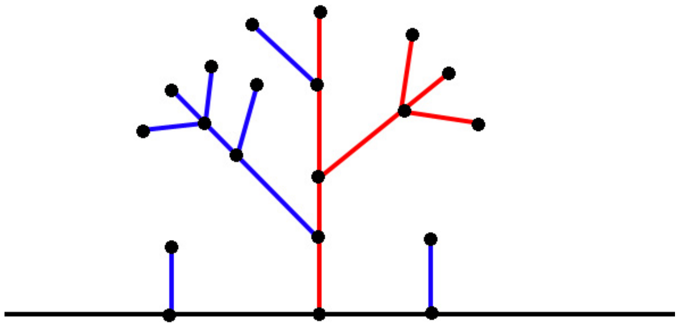


Figure 1. A sample game of Red-Blue Hackenbush.

Red-Blue Hackenbush is a two-person game – let's call these two people Left and Right – best played on a chalkboard. We begin by drawing a picture like Figure 1, made of Red and Blue lines, on which to play on. After this is done, Left and Right alternate turns, each erasing a (suggestively alliterative) Blue or Red edge from the graph, as well as any edges that are no longer connected to the ground (the thick black line at the base of the picture.) The game ends when one of our two players no longer has any edges he can cut – this player is the "loser" of a game of Hackenbush, and the other is (logically) the winner.

For example, in the picture below, the Right player is at a clear advantage – his shrub has 10 edges to Left's collective 6, and if the Right player prunes his shrubbery by moving from the top down, he can ensure that there will be at least $10 - 6 = 4$ edges left at the end of the game for him. Left can make sure that Right plays at least this well by being careful with how he trims his bushes as well.

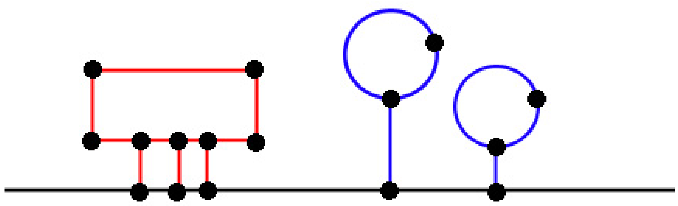


Figure 2. A shrubbery!

But what about a slightly less clear-cut picture? In Figure 3, both players have completely identical plants to prune, so who wins? Well: if Left starts, he will have to cut some edge of his picture. Right then can adopt a **mimicking strategy** and simply copy whatever Left has just done on his own picture, and thus be assured of always having a move left before the game ends. So Left will lose this match if he goes first – but if Right goes first, Left can also use this mimicking strategy and thus ensure that *Right* loses this match. We call these games where the first player to move loses **zero games**.

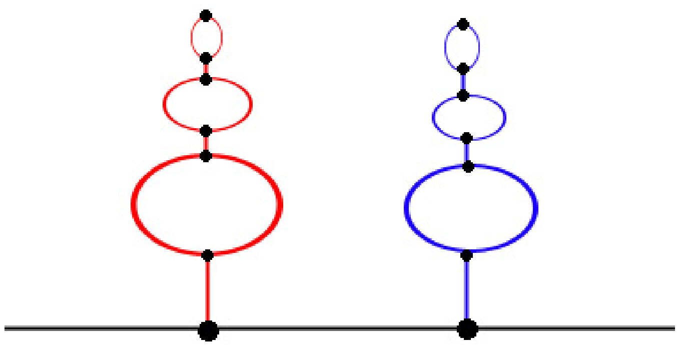
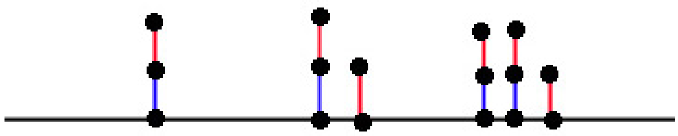


Figure 3. Another shrubbery!

So we're beginning to understand a bit about Hackenbush works – if Right and Left are both chivalrous players and have agreed to keep their colors on separate sides of their picture, we can look at any game and tell who will win by simply counting up the Blue edges and subtracting the number of Red edges. If the number is positive, left will win; if it is negative, Right will win; and if it is 0, the second player to go will win. Fortunately for us, things can get much more interesting than this, as the following picture demonstrates:



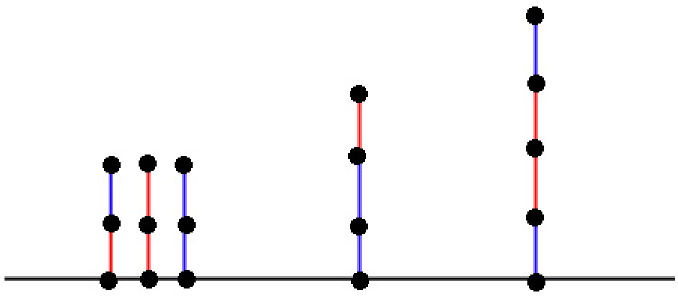
(a) (b) (c)

Figure 4. Fractional moves?

In Figure 4(a), it's fairly obvious that Left will win no matter what, as he can either simply cut his edge first and deprive Right of his Red edge to chop, or wait for Right to go first, cut his sole Blue edge and leave Right again without a move. But this doesn't fit into our simple hierarchy of games we've played so far – while Right is always winning, it isn't necessarily obvious how well he is doing so. So: by how much is Right winning?

As Figure 4(b) demonstrates, it's not by a whole move – for if we try to "balance" the game by adding an extra Red edge to the game (i.e. a 1-point advantage to Right), we find that now Right is winning every game: if Right goes first, it takes its Red edge that sits on top of the Blue edge and hands a **zero game** to Left (which ensures that Left will lose,) and if Left goes first, he must chop his sole Blue edge, which leaves Right a spare edge to cut and the victory. So this odd stack is worth some fractional amount of moves to Left **between** 0 and 1!

In fact, as Figure 4(c) shows, this is worth in fact $\frac{1}{2}$ of a move, as **two** copies of this move stuck together coupled with a single Red edge yields a zero game: if Left goes first, he shears one of the stacks, to which Right responds to by removing the top of another stack, returning a zero game and defeat to Left. Similarly, if Right goes first, he can take either a Red edge from atop a stack or from his spare edge; if the first, Left responds by taking a Blue edge from the stack that Right has not touched, handing a zero game back to Right and giving Left the win; if the second, Left simply takes a Blue edge from either stack, leaving the fractional game of 4(a) which Left is guaranteed to win no matter what happens. So, as the second player to go wins this game in 4(c), it is in fact a zero game, and so we can think of 4(a) as being worth $\frac{1}{2}$ of a move to Left.



(a) (b) (c)

Figure 5. Fractional moves!

Similar constructions can yield other fractional move-advantages: the reader is invited to verify before proceeding further that the three pictures in Figure 5 are worth $\frac{-1}{2}$, $\frac{1}{4}$, and $\frac{3}{8}$.

So: this far, it seems like our progress has been haphazard. We can now play some versions of Hackenbush fairly well, and can look at a few pictures and tell which side will win (given two players who know what they're doing.) A more formal approach might be easier on the brain, though: so let's introduce some notation. For any of our **games** G, we write

$$\{l_0, l_1, l_2, \ldots | r_0, r_1, r_2, \ldots\}$$

for a picture from which Left can move to a picture worth $l_0, l_1, l_2, \ldots$ (i.e. can move to a picture worth l_0, l_1,, or $l_2 \ldots$ moves to Left), and where Right can move to a picture worth $l_0, l_1, l_2, \ldots$ to them. (For sanity's sake, we denote a 1-turn advantage to Left as being 1, and a 1-turn advantage to Right as -1.)

In this notation, we get the integers expressed as follows:

$$0 = \{\ |\ \}, 1 = \{0|\ \}, 2 = \{1|\ \}, \ldots, -1 = \{\ |0\}, -2 = \{\ | -1\}, \ldots$$

Returning to our earlier games, we find that Figure 2 is $\{-5, -7| -3\}$, Figure 3 is $\{-1, -3, -6, -9|1, 3, 6, 9\}$, and Figure 4(a) is $\{0|1\}$. For brevity's sake, we will sometimes omit all but the largest value for Left's options, and all but the smallest value from Right's options, as we can assume that each player is playing in their own best interest – so Figure 3 can be simplified to $\{-1|1\}$, for example.

A natural question then arises: since this notation describes a game just as well as an actual picture does (as both describe the optimal moves for our players), how can we turn something like $\{a, b, c, \ldots |x, y, z, \ldots\}$ into a number?

To do this, we need something called the **Simplicity Rule**.

Let's begin with a few definitions.

Definition. *For a game G with options (i.e possible values)*

$$\{l_0, l_1, l_2, \ldots | r_0, r_1, r_2, \ldots\},$$

if all of the l_i, r_j are numbers, then the number x ***fits*** *if it is strictly greater than all of the l_i and strictly less than all of the r_i. A number x that fits is called the* ***simplest number*** *that fits if none of its* ***options*** *fit – i.e. if $x = \{a|b\}$, neither a nor b should fit into G.*

So, for Figure 5(a) $= \{-5/2, -3/2, -1|0, 1, 2\}$, the numbers $\frac{3}{4}$, $\frac{9}{16}$, and $\frac{1}{2}$ all fit, but only $\frac{-1}{2}$ is the simplest number to fit, as its options $\{-1|0\}$ do not fit into 5(a).

This leads us to the actual statement of the Simplicity Rule:

Rule. *The Simplicity Rule. A game G with options (i.e possible values)*

$$\{l_0, l_1, l_2, \ldots | r_0, r_1, r_2, \ldots\},$$

is equal to x, where x is the simplest number that fits into G.

Proof. First, observe that we can "add" games in a very natural way – if we have two Hackenbush pictures, we can just combine them by allowing Right and Left to play on either picture on their turns. For example, we can simply play on all of Figure 4 as opposed to breaking it up. As well, when we do so, the value of any such sum of games is simply the sum of their individual parts – for, since the pictures are all indivually disconnected from each other, moves in one picture do not affect the others. We have been using this principle quite liberally this far through the paper; if the reader is not convinced, we recommend working through some sample games to build intuition.

Similarly, we can create the inverse of a game $H = \{l|r\}$ by adding the game $-H := \{-r| - l\}$ – this is obviously a zero game, as in the game $H + -H$, if Left leads with either l or $-r$, Right can respond with either $-l$ or r to give Left a zero-game and ensure that Right wins, and if Right leads, Left can adopt a similar strategy to win as well. So $H + -H = 0$.

So: let us return to the question at hand. Without any loss of generality, we can write the game in question G as $\{a|b\}$ by simply ignoring all of the "bad" moves available to either player. Let $x = \{c|d\}, c > a, d < b$ be the simplest number that fits into G. Then $-x = \{-d| - c\}$, as we just showed, and we can then play on the Hackenbush game $G + -x$.

If Left leads, he can move to either $a + \{-d| - c\}$ or $\{a|b\} - d$, to which Right responds with his only possible move to make the game either $a - c$ or $b - d$. Both of these quantities are ≤ 0, as $c > a$ and $d < b$, so Right will necessarily win this game if Left starts. However, if Right starts, a similar argument shows that Left will win, so $G + -x$ is consequently a zero game.

Therefore, $G + -x = 0$, and so $G = x$. □

Finding these "simplest numbers" can be difficult at times: a picture from Berlekamp, Conway and Guy's **Winning Ways for your Mathematical Plays** is partially reproduced here in order to show how one determines which numbers are the simplest. To find the number for any game, simply traverse the tree below until you hit the simplest number that fits into your game, starting from the top (as { | } can be thought of as the "simplest" game.)

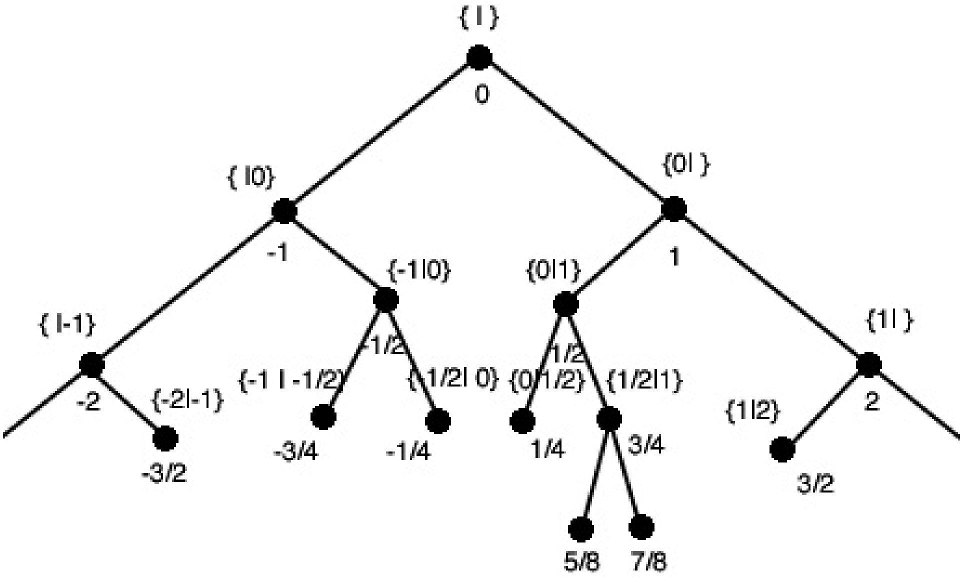


Figure 6. A simplest number tree.

By inductively repeating this process, we can assign a number to any Red-Blue Hackenbush picture, and thus know the winner of any game before it begins! Look at the house on the next page for an example – while studying the building itself can seem a bit confusing, we can simply break it down into individual cases, consider these, and then use the Simplicity Rule to get a value. Skeptical readers are invited to verify that Figure 7 is in fact worth -1 move by playing it with a lone Blue edge on the ground to balance, and seeing that this is a zero game.

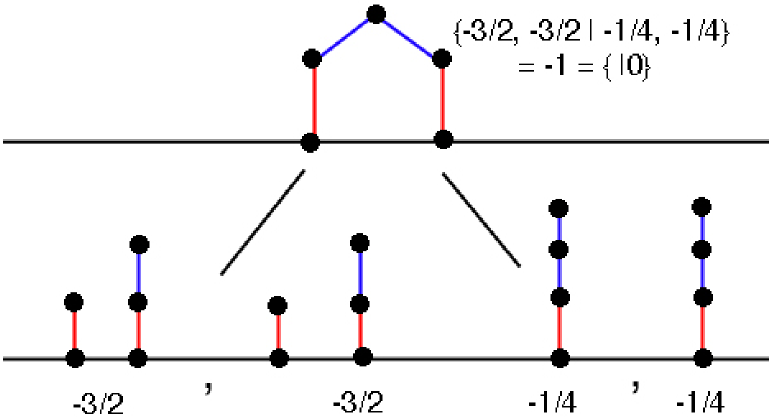


Figure 7. Burning Down the House.

2 But What If We're Colorblind?

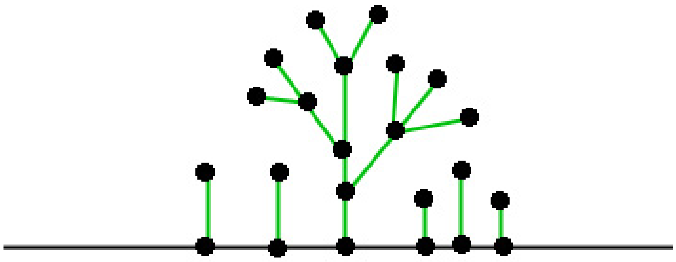


Figure 8. A Green Hackenbush Picture.

We've got a pretty good grasp on plain ol' Red-Blue Hackenbush at this point – so, let's try extending our newfound skills to a variation on the same game, called **Green Hackenbush** (or **Monochromatic Hackenbush**). In this game, we play with pictures made up of only Green edges, and we allow these edges to be cut by either player. This may seem to be a simpler version of the game we've been playing before – but some interesting complications arise when we study games further. Take, for example, the extremely simple game in Figure 9:

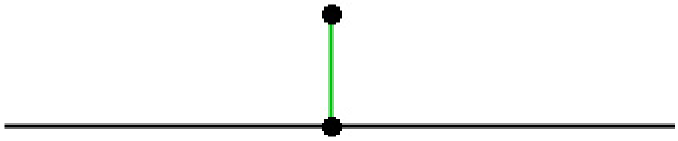


Figure 9. Thoreau's Ideal Garden.

Clearly, if Left goes first, he will instantly cut the stalk and thus win the game; conversely, if Right goes first, he will do the same and win the game as well. This situation, however, is one unlike any we've seen in Red-Blue Hackenbush, as the first player to go **wins**. Call these games **fuzzy games**.

What if we played a game with two copies of Figure 9? Clearly whoever would have to go first would have to cut one of the stalks, leaving the second player to simply cut the other and win the game – thus, the sum of two 1-stalk games would be a zero game. Similarly, by using a **mimicking strategy** as discussed before, we can see that the sum of two identical stalks of height n will be zero as well. Yet, it bears noting that the sum of two fuzzy games is not always a zero game, as Figure 10 illustrates: whichever player goes first can simply reduce the 2-stack to a 1-stack, thus turning the game into a zero game and ensuring that they win. So here, the sum of two fuzzy games is another fuzzy game! Curious . . .

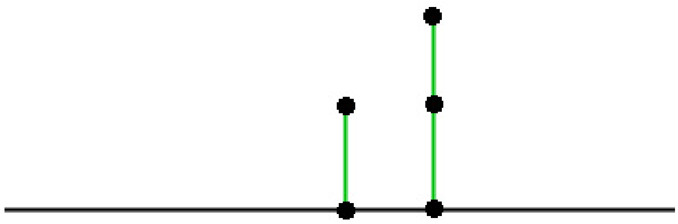


Figure 10. Adding Fuzzy Games.

How **do** we add these games, then? If we restrict ourselves to playing with bamboo stalks – i.e. just stacks of Green edges of various heights – it turns out that we are simply playing the game of Nim, where players first arrange stacks of coins on a table, and play by taking any amount of coins from any one of the heaps. Just as in Hackenbush, the last player able to move in such a way is the winner. For such games, we denote a stack (or heap) of size n by writing $*n$. So adding these simple games of Hackenbush amounts to simply knowing how to play Nim – and this is something that (provided a handy rule!) we can easily do.

First, note that any Nim-heap $*x$ has the options

$$\{*(x-1), *(x-2), *(x-3)\ldots, 0| *(x-1), *(x-2), *(x-3)\ldots, 0\},$$

as either player can reduce a heap of size x to one of a lower size, and both players have the same options. So:

Rule. *The Minimal-Excluded (Mex) Rule. If a game G has options*

$$\{*a_0, *a_1, *a_2, \ldots | *a_0, *a_1, *a_2, \ldots\},$$

with all of the same Nim-values available as options to either player, then the game G can itself be regarded as a Nim-heap of value $*x$, where x is the smallest whole number not equal to any of the a_i.

Proof. Well, suppose some player is playing the sum of games $G+H+K+\ldots$ and has a winning strategy in the game $*x+H+K+\ldots$. If he ever wants to move to any value in $*x$, this is obviously available to him, as G has every option that $*x$ has (otherwise, there would be a value smaller than x in G, and he would have not used the Mex rule correctly.) Furthermore, if his opponent moves to any option of $*x$, this is obviously not going to interfere with our winning player's plans, as he was already counting on G acting like $*x$ anyways.

So the only thing that can go wrong is if our player's opponent moves G to one of its options $*a_i$ that is **greater** than x. But if this happens, our winning player can simply reduce the Nim-heap to one of height x, as since a_i is bigger than x, $*a_i$ must have $*x$ as an option. This reduces the game to $*x+H+K+\ldots$ in fact, and our player will in fact win. Thus, for all intents and purposes, G is worth the same in any sum of games as a Nim-heap of size x, x being the smallest whole number not equal to any of the options of G. □

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	0	3	2	5	4	7	6	9
2	2	3	0	1	6	7	4	5	10
3	3	2	1	0	7	6	5	4	11
4	4	5	6	7	0	1	2	3	12
5	5	4	7	6	1	0	3	2	13
6	6	7	4	5	2	3	0	1	14
7	7	6	5	4	3	2	1	0	15
8	8	9	10	11	12	13	14	15	0

Figure 11. A Table of Nimbers.

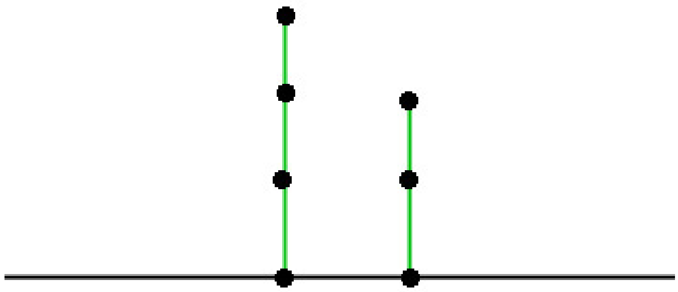


Figure 12. Nim-addition!

From this, we can carefully construct a Nim-addition table. To use this, we find our nim-heaps on the column and rows, count as options each of the nim-values we cross while moving to the square where the row and column intersect, and apply the Mex rule to those options – consequently Figure 12 is equivalent to a Nim-heap of height 1, as 1 is the smallest number that is not included in $3, 2$ or $3, 2, 0$. But constructing such a table through this inductive process is **slow!** – we want a faster way to determine the sum of a pair of "nimbers." Fortunately, one exists:

Rule. *The Nim-Addition Rule. For any two nimbers $*x$ and $*y$, their nim-sum $*x + *y$ is the number z that we get by writing x and y out as a sum of various powers of two (so, for example, $8 = 2^3, 7 = 2^0 + 2^1 + 2^2, 17 = 2^4 + 2^0 \ldots,$) cancelling any powers of two that appear twice, and summing what remains normally.*

In this fashion, we would conclude that

$$*4 + *7 = *(2^2) + *(2^0 + 2^1 + 2^2) = *(2^0 + 2^1) = *3$$

which is, in fact, exactly what we get with our table. (More computer-science minded people can think of this as simply writing any two heaps out in binary and then XOR-ing these two quantities together – these two actions are in fact the same.) The proof that this rule is relatively straightforward:

Proof. First, notice that $*1 + *k$ is equal to $k + 1$ if k is even, or $k - 1$ if k is odd, as applying the Mex principle and induction to our table shows – it's obviously true for $*1 + *0$ and $1 + *1$, and using the Mex rule shows that it will hold for all future numbers. Call this the Parity Rule for future reference.

So. We begin by proving the weaker statement that

$$(\forall n, k \in \mathbb{N}) \text{ such that } (n < 2^k), (*n + *2^k = *(n + 2^k).$$

If we can prove this statement, it is obvious that we can write any nimber as the sum of a variety of nim-stacks of heights equal to various powers of 2 – once that is done for any two nimbers $*x$ and $*y$, we can cancel any doubled powers of two by our earlier observation that the sum of any two Nim-heaps of the same height are just a zero game, and we can conclude our proof.

We proceed by induction. For $k = 1$, it is obvious that this statement holds, as our Nim-table constructed earlier shows that $*1 + *2 = *3$. So: given that for some $k \in \mathbb{N}$,

$$(\forall n, l \in \mathbb{N}) \text{ such that } (n < 2^l, l < k), (*n + *2^l = *(n + 2^l),$$

can we show that

$$(\forall n \in \mathbb{N}) \text{ such that } (n < 2^k), (*n + *2^k = *(n + 2^k)?$$

Well, as we're trying to prove a fact relating to an addition table, a second use of induction is the most intuitive step we can take at this juncture. It's fairly obvious that $*1 + *2^k = *(1 + 2^k$, as this is simply a special case of the Parity Rule we noticed earlier, so: we just need to show that if

$$(\forall m \in \mathbb{N}, m \leq n) \quad *n + *2^k = *(n + 2^k),$$

$$(*(n+1) + *2^k = *(n + 1 + 2^k)$$

is true. If n is odd, then $*(n + 1) = *(n - 1)$ by the Parity Rule, and so we are done by our inductive hypothesis: so it suffices to simply check that this holds if n is even.

Again, by the Parity Rule, since n is even, $*(n + 1) = *n + *1$, and so

$$(*(n+1) + *2^k = *n + *1 + *2^k$$

But this is just

$$*1 + *(n + 2^k)$$

by the inductive hypothesis, and since n and 2^k are both even, their sum is even, and we can thus use the Parity Rule one last time to conclude that this is just

$$*(1 + n + 2^k).$$

This concludes all of our inductive steps, allowing us to conclude that we can write any nimber $*x$ as the sum of nimbers of heights equal to powers of 2, and thus that our Nim-Addition Rule holds. □

We now completely understand any bamboo groves in Green Hackenbush through this principle; the question now, however, is how can we evaluate **any** given picture? We could use the methods of Red-Blue Hackenbush of just reducing all pictures into a series of smaller ones – but this is tedious! We again want a faster way.

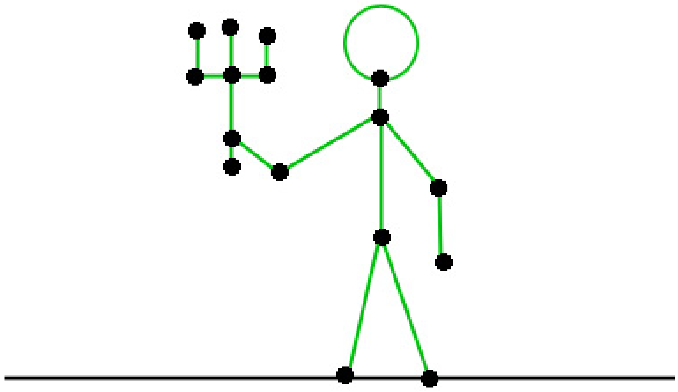
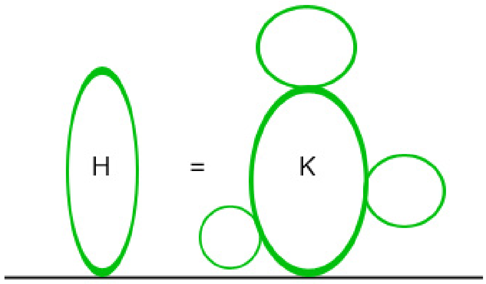


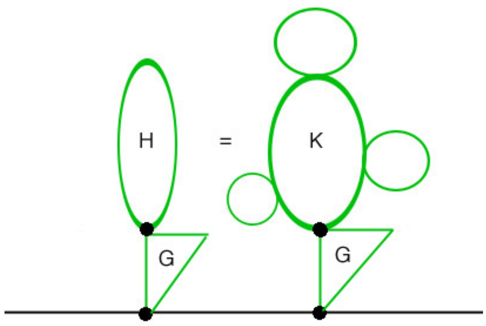
Figure 13. Mr. Green, with a Candlestick, in the Hall.

To accomplish this, we need to introduce two concepts, the first of which we will call the Colon Rule.

Rule. *The Colon Rule. For two Green Hackenbush pictures H, K,*
If



then



Figures 14 and 15. The Colon Rule.

Proof. To put it more formally, suppose we have a picture G with some node, a, onto which we "stick" a picture H. Then the claim is that this new picture – call it $G \cdot H_a$ – has the exact same value as $G \cdot K_a$ for **any** picture K that has the same value as H. (Notation such as this is why pictures are very useful.)

To see this, simply play Green Hackenbush on the sum of these two pictures, $G \cdot H_a + G \cdot K_a$. If these two games are in fact equal, then their sum should be a zero game, and so there must be a strategy for the second player to win. So, let the first player make any move in this picture. If they move in the G component of either picture, then there is trivially a corresponding move for the second player in the G component of the second picture. Otherwise, their move must lie in the H or the K component of one of these pictures, and so reduces one of these pictures to some value $*x$. But since $H = K$, we can have our second player simply reduce the other sub-picture to the same $*x$, and so the second player again has a move.

This strategy of **mimicking** Player 1's moves can be continued through the entire game by player 2, as his actions always return the picture to the form of $G' \cdot H'_a + G' \cdot K'_a$, for some games G', H', K' – so player 2 will never be without a move, and so must win the game. Hence, $G \cdot H_a = G \cdot K_a$, and so the Colon Rule holds. □

This allows us to find the value of the tiny forest in Figure 8 from the start of this section: by applying the Colon Rule, we can reduce the original tree to

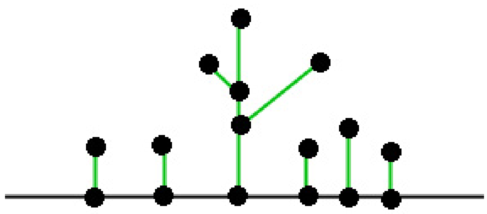


Figure 16. Clear-Cutting a Forest.

by replacing the pairs of two branches with nothing, and the triple of branches with 1; repeated iterations show us that the whole forest is equivalent to a Nim-heap of value $*0$.

This gives us a good deal of power: but how would we evaluate a sinister character like Figure 13? His candlestick is easy enough to break down, as is much of his body – but it is by no means apparent what to do about his legs, which cannot be reduced by the Colon Rule. So, to complete our study of Green Hackenbush, we need to introduce a few more concepts and one last rule.

Definition. *A **cycle** in a game of Green Hackenbush is a set of edges that forms a loop in a picture of Green Hackenbush – for the purposes of our definition, we count the ground as being one node.*

*This allows us to define **Fusion**, which is performed in the following way: two nodes in a cycle are **fused** by taking them and sticking them together, bending any edges that join the two into a loop at the new node.*

Rule. *The Fusion Principle. In any picture of Green Hackenbush, fusing any nodes together will never change the value of the game.*

Proof. Well, suppose not: then there must exist a picture such that some of its nodes cannot be fused. Pick the picture amongst these counter-examples with the smallest amount of edges (say, n), and amongst all of the non-fuseable pictures made of n edges, pick one – call it G – with the smallest amount of nodes. This gives us a picture G for any two nodes in the picture, we cannot fuse any of them, as this would yield a picture with less nodes and the same amount of edges, which violates the way we chose G.

We begin by observing a number of properties that G must possess.

1. G can only have one node that touches the ground. This is fairly obvious, as fusing all of the ground nodes on any picture together never changes how the picture works.

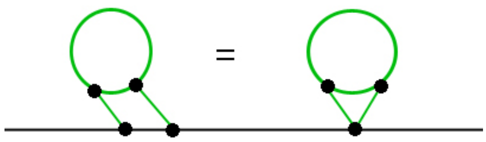


Figure 17. Fusing Ground Nodes.

2. G cannot contain any pair of nodes x, y, such that three or more distinct "paths" (i.e. sets of connected edges) connect x and y. This is because if G did so, then we could create a new game H by fusing x and y, which would have to have a different value than G (because we chose G such that no two nodes could be fused without changing its value.) So G and H are equivalent to 2 Nim-heaps of different values: therefore, there exists a winning move for the first player to move in $G + H$, as the sum of two different Nim-heaps is itself a non-zero Nim-heap(and thus a **fuzzy game**.)

   So, when the first player cuts an edge in either G or H, have the second player cut the corresponding edge in whichever picture the first person did not cut. This yields the two new games G' and H', each with $n-1$ edges in them – so we can fuse all of the nodes in each picture without changing the image. As well, since a and b started with three paths connecting them, there is still a cycle that loops them together in either G' or H', as we have deleted at most 1 path of the three available to us – so after fusion, these two graphs are identical. Therefore their sum is a zero game, and our first player will lose – thus demonstrating that the original G and H were in fact worth the same value.

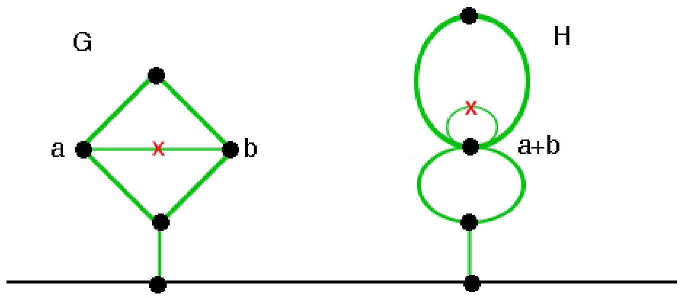


Figure 18. No Three Paths.

3. No cycle in G can exclude the ground. This can be seen by taking some such picture G with such a cycle C, and creating the picture G' by cutting each edge in this cycle. G' can obviously contain only one node from C, as if it contained two, that would imply that those two nodes were connected by three paths (the two necessary to create the cycle C and the one which connects them in G.) So G' contains only one node of G, and so C is only connected to G by one point – so we can apply the Colon Rule to C and whatever is attached to C. Since this picture has less nodes than G does, we can fuse everything on it and simply stick it back onto G – but this violates our assumption that we couldn't fuse anything in G. So no cycle in G can skip the ground.

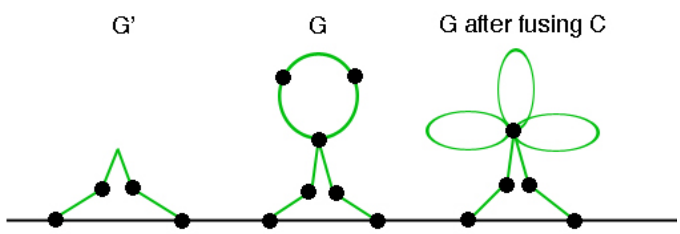


Figure 19. Cannot Avoid the Ground!

4. G contains only one such cycle that uses the ground, as otherwise it would be the sum of two smaller pictures, which we could individually apply the Fusion Principle to, or these two pictures would have to be connected in some third way, which would violate our second observation.

Figure 20. Tearing the Earth Asunder.

So G must look something like an arch with “things” hanging off of its various nodes – by the Colon Principle and by the minimal nature of our arch, we can apply fusion to each of these individual “things” and turn them all into strings. If the Fusion Principle holds, the value of any such arch with n edges making up its body is the Nim-sum of the strings that hang off of the arch $+1$, if n is odd, or $+0$ if n is even, because fusing the arch is just lumping all of the strings together onto the same node along with n single loops, which are each equivalent to a string of length 1.

If n is even, then the game in Figure 2, by our theory, must be a zero game. So: suppose the first player makes any move in any of the strings. Then the second player can simply respond with a move in the corresponding string in the grass or on the bridge, we can apply the Fusion Principle to the resulting smaller picture, and find out that it is, in fact, a zero game. So the first player can only cut an edge in the bridge – but this will invariably lead to a game with nonzero value, as it will leave an odd number of edges in the picture, and (as we showed earlier) Nim-addition respects parity. So we can use the Colon Principle on this smaller picture, reduce it to the equivalent of a Nim-string of nonzero length, and have the second player make the appropriate move to turn the picture into a zero game, and ensure that he wins. So Fusion works on a even bridge.

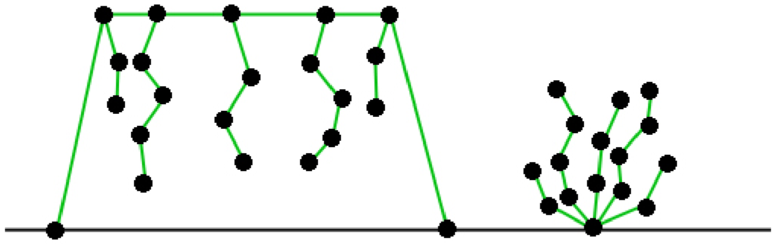


Figure 21. An Even Vault.

If n is odd, then we can apply a similar set of arguments using Figure 22 – if the first player moves in the strings of either side, we have the second player move in the strings of the other side, use the Fusion Principle and are done. Conversely, if they move in the bridge, they have to be leaving some sort of odd stack left, as this leaves an even number of edges in the bridge, 2· the number of edges in the strings, and the one extra edge put out to cancel – as Nim-addition (again) respects parity, applying the Colon Principle to this new picture shows that it's equivalent to a Nim-heap of odd height. So since none of the first player's moves are to Nim-heaps of value 0, we can use the **Mex Rule** to show that this whole picture is in fact a Nim-heap of value 0.

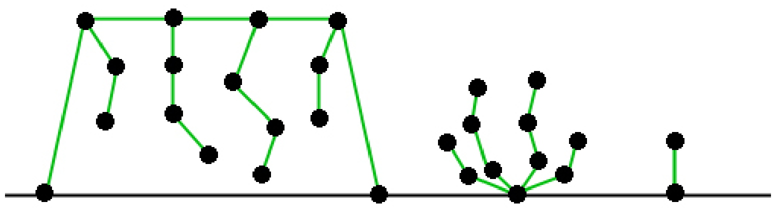


Figure 22. An Odd Overpass.

So the Fusion Principle holds.

□

Using these tools, we can now completely evaluate any Green Hackenbush picture – the reader is invited to verify, for example, that Figure 15 is equivalent to a Nim-string of height 6, Figure 18 to a string of height 2, and Figure 19 to a string of height 1.

3 R-G-B Hackenbush?

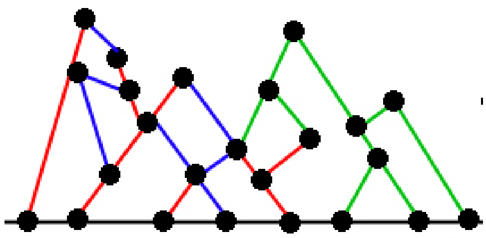


Figure 23. Hackenbush Himalayas.

We're pretty decent at Red-Blue Hackenbush, and have completely mastered Green Hackenbush by this point – so it only makes sense to combine two games we know how to play into one, right? Red-Blue-Green Hackenbush (or Hackenbush Hotchpotch) is played on a picture containing Red, Blue, and Green edges, with the Blue edges only touchable by the Left player, the Red edges only touchable by the Right player, and the Green edges available to Either player.

A complex picture like the one above, however, is a bit beyond us at the moment. So, let's restrict our pictures to something a bit more down to earth – a **flower garden**.

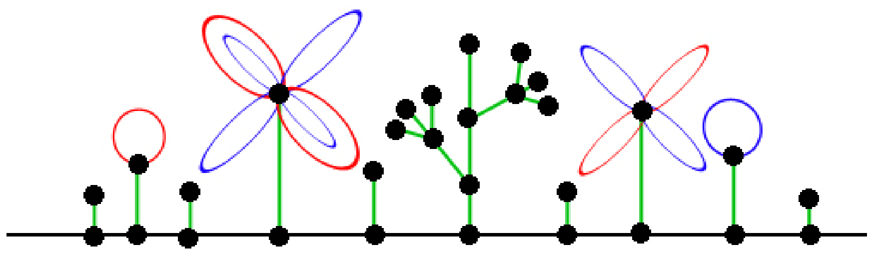


Figure 24. A Flower Garden.

Flower gardens are made of a collection of completely Green Hackenbush pictures and **flowers**, which are made of a **stalk** of Green edges topped by a **crown** of Blue or Red petals. By the Colon Rule, any flowers with r Red petals and b Blue petals are equivalent to a flower of $b - r$ Blue petals (or $r - b$ Red petals, equivalently) – so we can always assume that the blossoms of flowers in our garden are strictly monochromatic. But how should two players tend to such a garden?

Well, let's start in the even simpler case where Left has one morning glory in a garden, while Red is bereft of any roses of their own. Then if Left goes first, he has two choices – he can either take a petal from his flower, or chop down some Greenery. If we ignore Left's petals, we can evaluate the whole garden as a Green Hackenbush picture, and assign it a nimber, $*n$ – if this nimber is $*0$, Left can simply pluck a petal, hand the zero game to Right, ignore his flower for the rest of the game and win handily. If the nimber is greater than 0, then he can simply reduce it to 0 by playing in the Greenery and never touch his flower at all, and win this way.

So, if either player has one flower, the other player has none, and the beflowered player has the first move, they will automatically win – similarly, if one player has two flowers to the other's 0, then that player will win no matter who goes first, as the flower-less player can only cut one flower at a time.

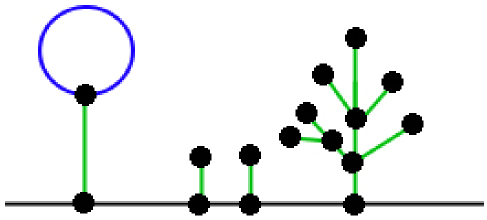


Figure 25. A Lonely Morning Glory.

From this, we can see that flowers outweigh the nimber value of the garden – if either player has a two-flower advantage, then they win automatically (given good play) and the weight of the Greenery is unimportant. From this, we can see that in a picture composed of flowers and Greenery, it makes much more sense for either player to cut down the other's flowers than to play with their own or the surrounding shrubberies, as cutting down the other player's flowers is the fastest way to try to get an advantage – as well, from this we can see that the shapes of flowers are relatively unimportant, as the only thing that matters is the quantity (except in the cases of single flowers). This leads us to the following definition and rule:

Definition. *We define the* ***atomic weight*** *of a Blue flower to be +1 and a Red to be flower -1; the* ***atomic weight*** *of a given garden is consequently defined as the sum of the atomic weights of all of the flowers in the garden.*

Rule. *The Flower Rule. If the atomic weight of a garden is* ≥ 2*, then (given perfect play) Left will always win any game of Hackenbush played in this garden; conversely, If the atomic weight of a garden is* ≤ -2*, Right will always win any game of Hackenbush played with this garden.*

Proof. We will prove the Flower Rule in the case of Left winning any garden of atomic weight ≥ 2; the proof for Right winning is completely identical. So.

Assume that the garden has $2 + n$ Blue flowers and k Red flowers in it, with $k \leq n$, and have Left chop a Red flower at each opportunity. Then after k exchanges there will be at least $2 + n - k$ Blue flowers left, and no Red flowers; since $2 + n - k \geq 2$, by the arguments forwarded above, it is obvious that Left will win. □

We can extend our study of flower gardens to wilder beasts, as well. A **parted jungle** is a generalization of a flower garden that looks like the tangle of Figure 26: essentially, they are an arbitrary Green Hackenbush picture with Red and Blue edges embedded in it such that no Red edges touch any Blue edges, and no Red or Blue edges touch the ground.

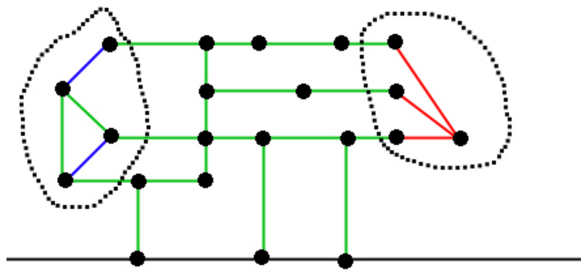


Figure 26. Parting the Jungle.

To understand how to hack through such jungles, we need to first have some definitions.

Definition. *A* ***track*** *from one node to another is a set of Green edges that connects these two nodes (treating the ground as all one node). A* ***flow*** *between two sets of nodes is a number of Green tracks from one set to another, with no tracks sharing any edges – a* ***flow*** *is* ***maximal*** *if it contains as many tracks as possible.*

Maximal flows are used to calculate the atomic weight of a jungle, and a rule we will present in a moment will explain exactly how to use them – but first, we should define how to create such a maximal flow.

First, take your jungle and identify the left and right sets by looking for clusters of Blue and Red edges. Some of your pictures may not be as nicely divided as Figures 26-29, but it is always possible to split the jungle into two clusters like we have done to our sample picture.

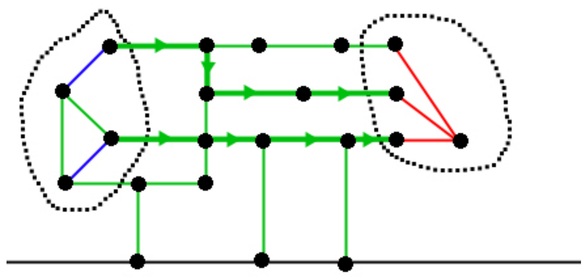


Figure 27. A First Attempt.

Once this is done, try and find as many tracks from the Blue cluster to the Red cluster as you can, taking care to not use any Green edges more than once, and labeling the direction that you traveled from the Blue cluster to the Red. This gives us the beginning of a maximal flow – but since it's possible that we've chosen our paths poorly, we're not quite done at this step.

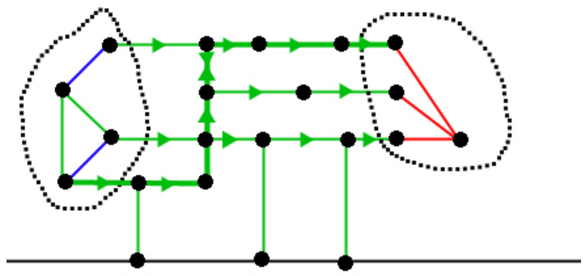


Figure 28. The Road Less Taken.

Then, take our Blue cluster, and try to reach the Red cluster on only edges we haven't used yet, or **backwards** along ones that we have. After this is done, we can simply delete any edges that were doublecrossed, and thus get a flow that is larger than the flow before. Repeating this process until it cannot be done any further will always generate a **maximal flow** from the Blue cluster to the Red cluster, as it will exhaust all possible paths from one to the other.

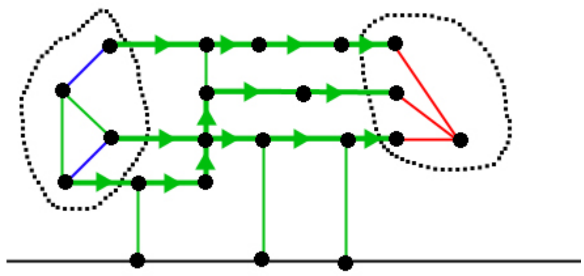


Figure 29. Putting Flow In Our Game.

Once we have a maximal flow, we can define an **enlargement** on our flow by trying to add as many tracks from one of the clusters to the ground as possible that do not conflict with our existing paths. It's somewhat obvious that this expansion can only add tracks to **one** of our clusters, as if it added a path from both of our clusters to the ground, this would in fact be a new path from the Blue cluster to the Red, and so would already be included in our maximal flow.

Now, with this maximal flow and its enlargement acquired, we can define the concept of **tinting** our nodes, and finally give the rule for hacking through parted jungles:

Definition. *A node is* ***tinted Blue*** *if it sits inside of our Blue cluster, or if we can reach this node by going along Green edges not in our flow, or backwards along ones that are: similarly, a node is* ***tinted Red*** *if it sits inside of the Red cluster, can be reached by Green edges not in the flow, or by going forwards along edges that are. As well, a node is* ***tinted Green*** *if you can reach it from the ground without using any edges carrying flow from the Blue cluster to the Red, and without going against the current of an edge carrying flow from either of the clusters to the ground. All nodes that are not tinted Red, Blue, or Green are defined as* ***untinted****.*

Rule. *If we have a parted jungle with a maximal flow and an enlargement of n tracks from the Blue cluster to the ground, the atomic weight of the jungle is n; conversely, if the enlargement consists of n tracks from the Red cluster to the ground, the atomic weight of the jungle is $-n$. If the atomic weight is ≥ 2, or Left has the first move and it is ≥ 1, Blue will always win; as well, if the atomic weight is ≤ -2, or Right has the first move and it is ≤ -1, Right will always win.*

Proof. The strategy for our players is very similar to the methods they used in the Flower Gardens; here, we will prove that Left will always win in parted jungles of atomic weight ≥ 2, or ≥ 1 if he has the move; the proof for Right's victory in other jungles is almost identical.

If Left has an advantage of ≥ 2, regardless of whether he goes first or second, he is guaranteed a jungle of atomic weight ≥ 1 when he gets to move first – so it is sufficient to prove that Left wins if he goes first on a jungle of atomic weight m, $m \geq 1$. Let the jungle have a maximal flow of n paths, and let Left, on each of his turns, cut an edge that crosses the boundary of Right's cluster. No matter what Right responds with, after n turns Right's cluster will be completely removed from the jungle, while at least some part of Left's cluster will remain, as there are $n + m$ edges that connect Left's cluster to other things.

Once this is accomplished, Left simply has to evaluate the jungle as a Green Hackenbush picture, ignoring his own edges. If the picture is equivalent to a Nim-heap of value 0, he trims one of his own edges, hands the zero game to Red and wins handily; if it is a Nim-heap of positive value $*x$, he ignores his own edges completely, reduces the whole picture to $*0$ through a move in the Green Hackenbush pieces, and wins again. Therefore, if the atomic weight of a jungle is $+1$, Blue will always win if he has the move, and so will always win if in a jungle of atomic weight $+2$. □

4 Conclusion

There are obviously many more cases left to study in Hackenbush Hotchpotch, many of which are provably NP-hard and some of which we lack any bounds at all with which to study the picture – this guide is merely a primer to the massive world of possible Hackenbush pictures that await. This paper is deeply indebted to Berlekamp, Conway and Guy's book **Winning Ways for your Mathematical Plays**, inside of which almost every result in this paper is written, along with hundreds of other beautiful games; interested readers should also peruse Conway's **On Numbers and Games**, which uses Hackenbush to set up the surreal number system, an alternative numerical system that includes the real numbers and Cantor's ordinal numbers, and of course the wonderful information-filled Internet.

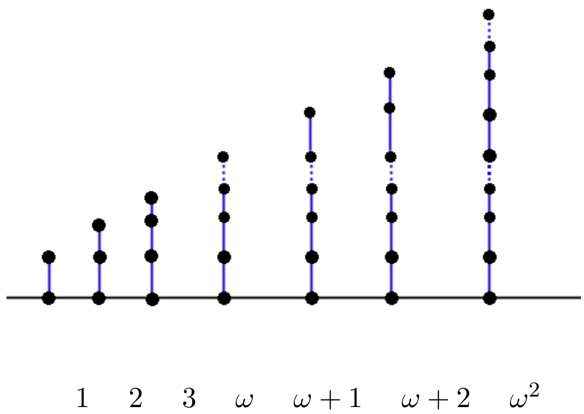


Figure 29. Counting to Infinity.

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